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Generic adjoints in comtrans algebras of bilinear spaces ☆

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Abstract

As a first step towards a general structure theory for comtrans algebras (modeled loosely on the Cartan theory for Lie algebras), this paper investigates comtrans algebras of bilinear spaces. Attention focuses on invariants associated with comtrans algebras, and the extent to which these invariants may serve to specify the algebras up to isomorphism within certain classes. Over fields whose characteristic differs from two, comtrans algebras of symmetric forms are determined up to isomorphism by the eigenvalues of generic adjoints, while comtrans algebras of symplectic forms are determined by the dimensions of maximal abelian subalgebras. Examples show that the multiplicity of zero as a root of the characteristic polynomial is generally independent of the dimension of a maximal abelian subalgebra.

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1. Introduction

Comtrans algebras are unital modules over a commutative ring R, equipped with two basic trilinear operations: a *commutator* $[x, y, z]$ satisfying the *left alternative identity*

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$$[x, x, y] = 0, \tag{1.1}$$

and a *translator* $\langle x, y, z\rangle$ satisfying the *Jacobi identity*

$$\langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0, \tag{1.2}$$

such that together the commutator and translator satisfy the *comtrans identity*

$$[x, y, x] = \langle x, y, x\rangle. \tag{1.3}$$

Comtrans algebras were introduced [17] in answer to a problem from differential geometry, asking for the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n + 1)$-web (cf. [2]). The role played by comtrans algebras is analogous to the role played by the Lie algebra of a Lie group. A comtrans algebra is said to be *abelian* if its commutator and translator are identically zero. Thus in essence, abelian comtrans algebras are just R-modules.

Comtrans algebras have been shown to arise in many different contexts [6,7,8,12,14,15,16,18]. For example, Shen and Smith [15] demonstrates how sets of rectangular matrices form comtrans algebras in natural fashion. A long-term goal of the research effort devoted to comtrans algebras is to develop a general structure theory for them, inspired by two classical applications of linear algebra: Frobenius' original approach[1] to the character theory of finite groups, and the Cartan theory for Lie algebras. Now in Lie algebra theory over a field of characteristic zero, the multiplicity of zero as a root of the characteristic polynomial gives the dimension of a Cartan subalgebra [9, Chapter III], [11, Theorem III.1]. Thus in developing the structure theory for comtrans algebras, two immediate questions arise:

1. Is there any analogous relationship between the multiplicity m of zero as a root of the characteristic polynomial (Definition 2.2), and the dimension d of a maximal abelian subalgebra?
2. To what extent do d and the roots of the characteristic polynomial serve to specify a comtrans algebra?

From this standpoint, the current paper examines comtrans algebras $\mathrm{CT}(E, \beta)$ over a field R furnished by *bilinear spaces* (E, β), finite-dimensional R-vector spaces E equipped with a bilinear form β [3, §5.1]. Such a comtrans algebra $\mathrm{CT}(E, \beta)$ has underlying module E. Its algebra structure is defined by

$$[x, y, z] = y\beta(x, z) - x\beta(y, z) \tag{1.4}$$

and

$$\langle x, y, z\rangle = y\beta(z, x) - x\beta(y, z) \tag{1.5}$$

[14]. For three-dimensional Euclidean space, (1.4) and (1.5) each coincide with the usual vector triple product.

Section 2 recalls the three kinds of adjoint map that feature in the representation theory of comtrans algebras as presented in [13]. Definition 2.2 then introduces the *characteristic endomorphisms*, the generic adjoints that form the key tools of our structure theory. Section 3 looks at the first two such adjoints, the *characteristic endomorphism* coming from the commutator and the *right characteristic endomorphism* coming from the translator. Theorem 3.4 shows that over fields whose characteristic differs from two, the comtrans algebras of symmetric forms are determined up to isomorphism by the eigenvalues of these endomorphisms. Section 4 considers

[1] Compare [1,4,10].

the remaining cases: the *left characteristic endomorphisms* coming from the translator, and all three characteristic endomorphisms for the comtrans algebras of symplectic spaces. It transpires (Theorem 4.2) that these endomorphisms are all nilpotent. However, for the comtrans algebra of a symplectic space, Proposition 4.3 specifies the dimension d of a maximal abelian subalgebra. Theorem 4.4 then shows that over fields whose characteristic differs from two, this invariant d, along with the dimension n, suffices to specify the comtrans algebra up to isomorphism within the class of all such algebras. Section 5 gives an answer to the first question posed, examining the relationship between the multiplicity of zero as a root of the characteristic polynomial and the dimension of maximal abelian subalgebras.

The results of this paper give a clear indication that the structure theory for comtrans algebras will be considerably more elaborate than the classical theory for Lie algebras. Given the broad scope of comtrans algebras, this is only to be expected.

For concepts and conventions of algebra that are not otherwise explained in the paper, readers are referred to [19].

2. Adjoint maps

The class $\mathfrak{CT}_R$ of all comtrans algebras over a ring R forms a variety in the sense of universal algebra, the class of all algebras satisfying a given set of identities. This variety becomes (the class of objects of) a bicomplete category whose morphisms are the homomorphisms between comtrans algebras (cf. Theorems IV 2.1.3 and 2.2.3 of [19]). For a member E of $\mathfrak{CT}_R$, let $E[X]$ denote the coproduct of E in $\mathfrak{CT}_R$ with the free $\mathfrak{CT}_R$-algebra on a singleton $\{X\}$. For x, y in E, there are R-module homomorphisms

$$K(x, y)\colon E[X] \to E[X]; \quad z \mapsto [z, x, y], \tag{2.1}$$

$$R(x, y)\colon E[X] \to E[X]; \quad z \mapsto \langle z, x, y\rangle, \tag{2.2}$$

and

$$L(x, y)\colon E[X] \to E[X]; \quad z \mapsto \langle y, x, z\rangle. \tag{2.3}$$

These endomorphisms of $E[X]$, or their restrictions to endomorphisms of E alone, are known as *adjoint maps*. (The terminology is analogous to that of Lie theory, and is not to be confused with adjoint operators.) The *universal enveloping algebra* $U(E)$ of E is the R-subalgebra of the endomorphism ring of the R-module $E[X]$ generated by

$$\{K(x, y), R(x, y), L(x, y) | x, y \in E\}$$

[13].

Proposition 2.1. *In the enveloping algebra $U(E)$ of a comtrans algebra E, one has*

$$K(x, x) - R(x, x) - L(x, x) = 0. \tag{2.4}$$

Proof. Apply the left hand side of (2.4) to an element z of $E[X]$ and simplify by consecutive use of (1.1), (1.3), and (1.2). □

The following definitions are modeled on the concept of the characteristic polynomial of a Lie algebra (cf. §III.1 of [9]). Towards the definition, suppose that R is a subring of a ring S.

The forgetful functor $\downarrow^S_R\colon \mathfrak{CT}_S \to \mathfrak{CT}_R$ preserves underlying sets. It thus possesses a left adjoint $\uparrow^S_R\colon \mathfrak{CT}_R \to \mathfrak{CT}_S$ described as *extension* to S (compare [19, IV, Corollary 3.4.8]).

Definition 2.2. Let R be a field, and let E be a comtrans algebra of finite dimension n over R. Let $\{X_1, \ldots, X_n\}$ be a set of n indeterminates over R, and let $\{e_1, \ldots, e_n\}$ be a basis for E over R. Then the *characteristic endomorphism* of the comtrans algebra E with respect to the basis $\{e_1, \ldots, e_n\}$ is the endomorphism

$$K(\mathbf{X}) := K(X_1e_1 + \cdots + X_ne_n, X_1e_1 + \cdots + X_ne_n) \tag{2.5}$$

of the extension of the comtrans algebra E to the field of rational functions over the set $\{X_1, \ldots, X_n\}$ of indeterminates. The *right characteristic endomorphism* of E with respect to the basis $\{e_1, \ldots, e_n\}$ is the endomorphism

$$R(\mathbf{X}) := R(X_1e_1 + \cdots + X_ne_n, X_1e_1 + \cdots + X_ne_n). \tag{2.6}$$

The *left characteristic endomorphism* of E with respect to the same basis is the endomorphism

$$L(\mathbf{X}) := L(X_1e_1 + \cdots + X_ne_n, X_1e_1 + \cdots + X_ne_n). \tag{2.7}$$

The *characteristic polynomial*, *right characteristic polynomial* and *left characteristic polynomial* of the comtrans algebra E with respect to the basis $\{e_1, \ldots, e_n\}$ are the respective characteristic polynomials of the endomorphisms (2.5)–(2.7).

3. Bilinear spaces

Let (E, β) be a bilinear space over a field R. Let $\{e_1, \ldots, e_n\}$ be a basis for E over R. Suppose that the matrix of β with respect to the basis $\{e_1, \ldots, e_n\}$ is B. For vectors $\mathbf{X} = [X_1, \ldots, X_n]$ and $\mathbf{Y} = [Y_1, \ldots, Y_n]$ of indeterminates, the *bilinear polynomial* of β with respect to $\{e_1, \ldots, e_n\}$ is

$$b(\mathbf{X}, \mathbf{Y}) = \sum_{1\leqslant i,j\leqslant n} B_{ij}X_iY_j$$

[3, §5.1]. The *quadratic polynomial* of β with respect to $\{e_1, \ldots, e_n\}$ is $b(\mathbf{X}) := b(\mathbf{X}, \mathbf{X})$ (compare [3, §5.3]). Recall that the bilinear form β on E is said to be *alternating* or *symplectic* iff $\beta(x, x) = 0$ for each element x of E. In that case, the space (E, β) is described as *alternating* or *symplectic*.

Lemma 3.1. *The quadratic polynomial $b(\mathbf{X})$ vanishes if and only if the bilinear form β is alternating.*

Proposition 3.2. *Let (E, β) be a bilinear space with basis $\{e_1, \ldots, e_n\}$. Take matrices of endomorphisms of E and its extensions with respect to this basis. Let $X_1, \ldots, X_n$ be indeterminates.*

(a) *The matrix of $K(\mathbf{X})$ is*

$$B[X_1, \ldots, X_n]^{\mathrm{T}}[X_1, \ldots, X_n] - b(\mathbf{X})I_n.$$

(b) *The matrix of $R(\mathbf{X})$ is*

$$B^{\mathrm{T}}[X_1, \ldots, X_n]^{\mathrm{T}}[X_1, \ldots, X_n] - b(\mathbf{X})I_n.$$

(c) *The matrix of* $L(\mathbf{X})$ *is*

$$(B - B^{\mathrm{T}})[X_1, \dots, X_n]^{\mathrm{T}}[X_1, \dots, X_n].$$

Proof. (a) By (1.4), for $1 \leqslant k \leqslant n$,

$$\begin{aligned} &e_k K\left(\sum_{i=1}^n X_i e_i, \sum_{j=1}^n X_j e_j\right) \\ &= \left(\sum_{i=1}^n X_i e_i\right)\beta\left(e_k, \sum_{j=1}^n X_j e_j\right) - e_k \beta\left(\sum_{i=1}^n X_i e_i, \sum_{j=1}^n X_j e_j\right) \\ &= \left(\sum_{i=1}^n X_i e_i\right)\sum_{j=1}^n B_{kj} X_j - e_k b(\mathbf{X}) \\ &= \sum_{l=1}^n \left[\left(\sum_{j=1}^n B_{kj} X_j\right) X_l - b(\mathbf{X})\delta_{kl}\right] e_l. \end{aligned}$$

(b) is obtained in similar fashion using (1.5).
(c) Apply Proposition 2.1 to (a) and (b). □

Proposition 3.3. *Let* (E, β) *be a bilinear space with basis* $\{e_1, \dots, e_n\}$. *Suppose that* β *is not alternating. Then both the characteristic endomorphism and the right characteristic endomorphism of* $\mathrm{CT}(E, \beta)$ *have zero as an eigenvalue of geometric multiplicity one, and the negated quadratic polynomial* $-b(\mathbf{X})$ *as an eigenvalue of geometric multiplicity* $n - 1$.

Proof. By Proposition 3.2(a) and (b), it is clear that the vector $[X_1, \dots, X_n]$ lies in the kernel of $K(\mathbf{X})$ and $R(\mathbf{X})$, so that zero has multiplicity not less than one as a root of each characteristic polynomial. On the other hand, the ranks of the matrices

$$B[X_1, \dots, X_n]^{\mathrm{T}}[X_1, \dots, X_n]$$

of $K(\mathbf{X}) + b(\mathbf{X})$ and

$$B^{\mathrm{T}}[X_1, \dots, X_n]^{\mathrm{T}}[X_1, \dots, X_n]$$

of $R(\mathbf{X}) + b(\mathbf{X})$ are at most one, so that the kernels of $K(\mathbf{X}) + b(\mathbf{X})$ and $R(\mathbf{X}) + b(\mathbf{X})$ have dimension at least $n - 1$. Since β is not alternating, Lemma 3.1 shows that the eigenvalues 0 and $-b(\mathbf{X})$ are distinct. Thus their geometric multiplicities both attain their respective minimum values of 1 and $n - 1$. □

Theorem 3.4. *Fix a field* R *that is not of characteristic two. Within the class of comtrans algebras* $\mathrm{CT}(E, \beta)$ *of bilinear spaces with symmetric forms* β *over* R, *a given comtrans algebra is determined up to isomorphism by its dimension and the roots of its characteristic polynomial.*

Proof. If E is one-dimensional, the comtrans algebra $\mathrm{CT}(E, \beta)$ is necessarily abelian. For any dimension, the algebra $\mathrm{CT}(E, \beta)$ is abelian if and only if zero is the only root of the characteristic polynomial. Otherwise, the negated quadratic polynomial $-b(\mathbf{X})$ is specified as the non-zero root of the characteristic polynomial, and since the characteristic of the ground field is not two, the bilinear space (E, β) is determined up to isomorphism as the space R^n with bilinear form

$$\beta\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} y_i e_j\right) = (b(x_i + y_i) - b(x_i) - b(y_i))/2,$$

$\{e_1, \ldots, e_n\}$ being the standard basis. Finally, note that isomorphic bilinear spaces have isomorphic comtrans algebras. □

4. Symplectic spaces

The preceding section avoided consideration of symplectic spaces. These spaces are examined in the current section.

Proposition 4.1. *In the comtrans algebra of a bilinear space, the left characteristic endomorphism is nilpotent.*

Proof. In the comtrans algebra of any bilinear space (E, β), the definition (1.5) yields

$$\begin{aligned}\langle x, x, \langle x, x, y\rangle\rangle &= \langle x, x, x\beta(y, x) - x\beta(x, y)\rangle \\ &= x\beta(x\beta(y, x) - x\beta(x, y), x) - x\beta(x, x\beta(y, x) - x\beta(x, y)) \\ &= x\beta(x, x)\beta(y, x) - x\beta(x, x)\beta(x, y) \\ &\quad - x\beta(x, x)\beta(y, x) + x\beta(x, x)\beta(x, y) = 0\end{aligned}$$

for elements x, y of E. Thus

$$yL(x, x)^2 = 0. \tag{4.1}$$

Now suppose that (E, β) is a bilinear space with basis $\{e_1, \ldots, e_n\}$ over a field R. Let S be the field $R(X_1, \ldots, X_n)$ of rational functions in indeterminates $X_1, \ldots, X_n$. The extension $\mathrm{CT}(E, \beta) \uparrow_R^S$ of $\mathrm{CT}(E, \beta)$ to S is the comtrans algebra of the bilinear space obtained by extending β to $S \otimes E$. Eq. (4.1) in $\mathrm{CT}(E, \beta) \uparrow_R^S$ then shows that the left characteristic endomorphism of $\mathrm{CT}(E, \beta)$ is nilpotent. □

Theorem 4.2. *Let (E, β) be a bilinear space with basis $\{e_1, \ldots, e_n\}$.*

(a) *The left characteristic endomorphism of* $\mathrm{CT}(E, \beta)$ *has zero as an eigenvalue of algebraic multiplicity n.*
(b) *Suppose that β is alternating. Then all three characteristic endomorphisms of* $\mathrm{CT}(E, \beta)$ *have zero as an eigenvalue of algebraic multiplicity n.*

Proof. Part (a) of the theorem follows immediately by Proposition 4.1. Now suppose that β is alternating. By Lemma 3.1, the quadratic polynomial vanishes. Moreover, the alternating matrix B of β with respect to $\{e_1, \ldots, e_n\}$ may be written in the form $B = C - C^{\mathrm{T}}$ for some matrix C. Proposition 3.2 then shows that the matrix of the characteristic endomorphism of $\mathrm{CT}(E, \beta)$ with respect to $\{e_1, \ldots, e_n\}$ has the same form as the matrix Λ of the left characteristic polynomial of $\mathrm{CT}(E, \gamma)$, for a bilinear form γ on E having matrix C with respect to $\{e_1, \ldots, e_n\}$. By part (a) of the theorem, the matrix Λ has zero as an eigenvalue of algebraic multiplicity n. A similar argument shows that the right characteristic polynomial of $\mathrm{CT}(E, \beta)$ also has zero as a root of multiplicity n. $\square$

Theorem 4.2 shows that for a symplectic space (E, β), the structure of $\mathrm{CT}(E, \beta)$ is not reflected in the eigenvalues of the characteristic endomorphisms. The dimensions of the maximal abelian subalgebras are more informative.

Proposition 4.3. *Let (E, β) be a symplectic space with basis $\{e_1, \ldots, e_n\}$ and rank r. Then the dimension of a maximal abelian subalgebra of $\mathrm{CT}(E, \beta)$ is $n - \frac{1}{2}r$.*

Proof. The Structure Theorem for alternating (or symplectic) spaces (e.g. [3, Theorem 5.2], [5, Satz II.9.6]) shows that without loss of generality, the matrix B of β with respect to $\{e_1, \ldots, e_n\}$ may be taken to be

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \mathrm{diag}(0, \ldots, 0) \tag{4.2}$$

with $\frac{1}{2}r$ summands having the form

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

of the matrix of a hyperbolic space. Then

$$\{e_1 + e_2, \ldots, e_{r-1} + e_r, e_{r+1}, \ldots, e_n\}$$

spans an abelian subalgebra of dimension $n - \frac{1}{2}r$.

Conversely, suppose that $\mathrm{CT}(E, \beta)$ has an abelian subalgebra V of dimension d. By [14, Theorem 3.6], it follows that $(V, \beta|_V)$ is an isotropic subspace of (E, β). By the Structure Theorem, the matrix B of β is of the form (4.2), so that (E, β) is the direct sum of a nondegenerate symplectic space (E_1, β_1) of dimension r and a trivial symplectic space (E_0, β_0) of dimension $n - r$. Then [5, Satz II.9.11] implies that $d \leqslant \frac{1}{2}r + (n - r)$. $\square$

Theorem 4.4. *Fix a field R that is not of characteristic two. Within the class of comtrans algebras of finite-dimensional symplectic spaces over R, a given comtrans algebra is determined up to isomorphism by the dimension n of the whole algebra and the dimension d of any maximal abelian subalgebra.*

Proof. Suppose that $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \gamma)$ are comtrans algebras of respective symplectic spaces (E, β) and (E, γ) with the same underlying vector space E of dimension n. Suppose that $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \gamma)$ have the same dimension d for their maximal abelian subalgebras. By Proposition 4.3, both β and γ have rank $2(n - d)$. Then by the Structure Theorem, the spaces (E, β) and (E, γ) are both equivalent to the space whose form has matrix (4.2). Since (E, β) and (E, γ) are equivalent, it follows that the algebras $\mathrm{CT}(E, \beta)$ and $\mathrm{CT}(E, \gamma)$ are isomorphic. $\square$

5. Abelian subalgebras

This section addresses the relationship between the multiplicity m of zero as a root of the characteristic polynomial, and the dimension d of a maximal abelian subalgebra. Recall that in classical Lie theory over a field of characteristic zero, the multiplicity of zero as a root of the characteristic polynomial is equal to the dimension of a Cartan subalgebra [9, Chapter III], [11, Theorem III.1]. The examples of this section exhibit both positive and negative results within the context of comtrans algebras of bilinear spaces. In Example 5.1, $m/d = 2$; in Example 5.2, $m = d = 1$; while in Example 5.3, $m/d \to 0$.

Example 5.1. For a nondegenerate symplectic space of (even) dimension n, Theorem 4.2(b) shows that $m = n$, while Proposition 4.3 shows that $d = n/2$. Thus $m/d = 2$.

Example 5.2. Over a field R that is not of characteristic two, consider a symmetric bilinear space (E, β) having no two-dimensional isotropic subspaces. Note that the form β is not alternating. By Proposition 3.3, zero has multiplicity one as a root of the characteristic polynomial. Then the dimension of a maximal abelian subalgebra of $\mathrm{CT}(E, \beta)$ is also one. Certainly any one-dimensional subspace of E forms an abelian subalgebra. Suppose that two elements x, y of E were to span a two-dimensional abelian subalgebra. Now by (1.4),

$$[x, y, x] = y\beta(x, x) - x\beta(y, x). \tag{5.1}$$

From the hypothesis on the isotropic subspaces, it follows that (5.1) is non-zero, contradicting the assumption.

Example 5.3. For a positive integer $n > 2$, consider the real space $\mathbb{R}^n$ equipped with the symmetric bilinear form whose matrix with respect to the standard basis $\{e_1, \ldots, e_n\}$ is $\mathrm{diag}(1, 0, 0, \ldots, 0)$. By Proposition 3.3, zero has multiplicity one as a root of the characteristic polynomial. On the other hand, the basis elements $e_2, \ldots, e_n$ span an $(n-1)$-dimensional abelian subalgebra. Thus as n tends to infinity, the multiplicity of zero as a root of the characteristic polynomial stays constant, while the dimensions of maximal abelian subalgebras grow linearly.

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